Cutoff for the quantum unitary Brownian motion

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I/ Cutoff

Fix a compact group G. We call <u>Lévy process</u> on G a family of G-valued random variables $(g_t)_{t\geq 0}$ (taking values in the same probability space) s.t. [random walk if indexed by $\mathbb{N}$]

(i) $\mathrm{Law}(g_{t+s}g_s^{-1})$ only depends on t.
(ii) $g_t \to g_0$ in probability as $t \to 0$.
(iii) $g_{t_2}g_{t_1}^{-1}, \dots, g_{t_n}g_{t_{n-1}}^{-1}$ are independent for any $0 \leq t_1 \leq \dots \leq t_n$.

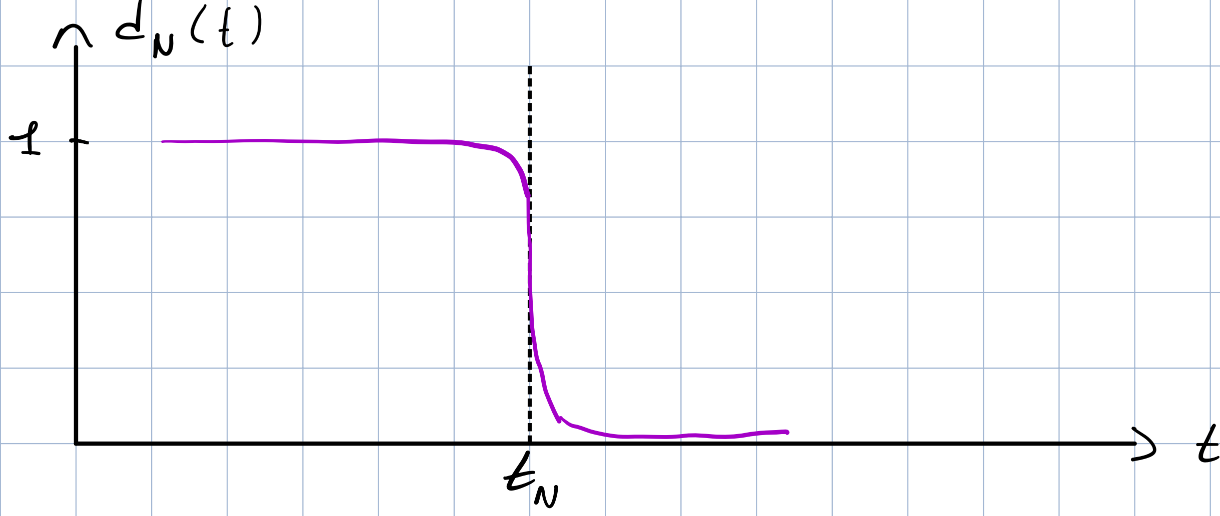
$$\mu_t := \mathrm{Law}(g_{t+s}g_s^{-1}) \qquad (\mu_t)_{t\geq 0}$$

(i) $\mu_0 = \delta_e$
(ii) $\mu_t \to \mu_0$ weakly as $t \to 0$
(iii) $\mu_{t+s} \to \mu_t * \mu_s$

Def. Let $(G_N, \mu^{(N)})_{N\in\mathbb{N}}$ be a family of compact groups each equipped with a Lévy process (or a random walk). We say that it exhibits <u>cutoff</u> at time t_N if

$$d_N(t_N(1-\varepsilon)) \xrightarrow[N\to\infty]{} 1 \qquad \& \qquad d_N(t_N(1+\varepsilon)) \xrightarrow[N\to\infty]{} 0, \qquad \varepsilon > 0$$

where $d_N(t) := d_{TV}(\mu_t^{(N)}, \mathrm{Haar}_N)$.



Some notable cutoff

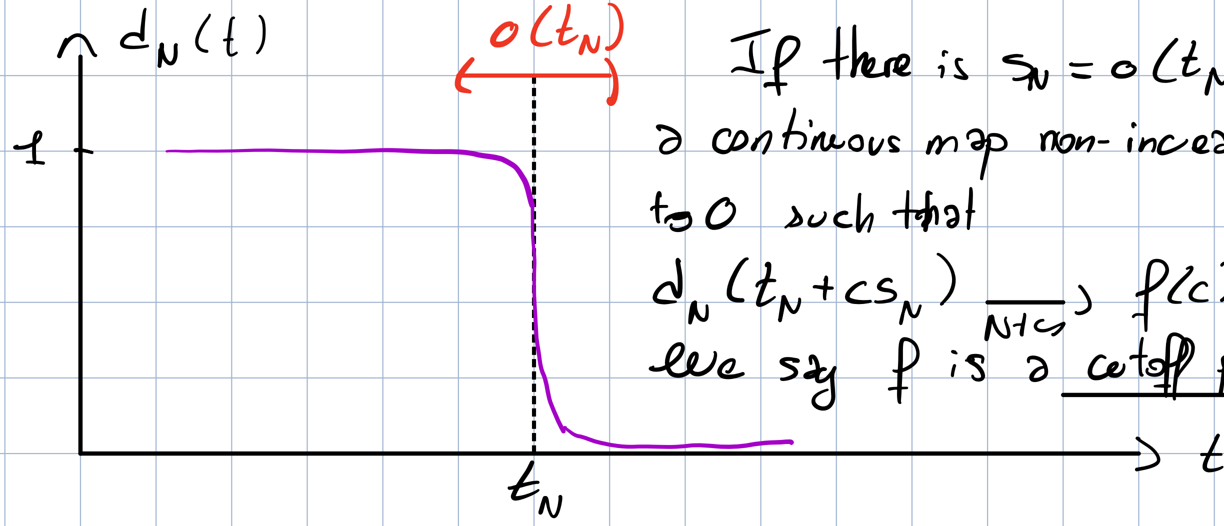
* Random transpositions on S_N: $t_N = \frac{1}{2} N \ln N$ (Diaconis-Shahshahani 80s)

* Brownian motion on Lie groups: $t_N = \alpha \ln N$ (Meliot 13)
  $\alpha \in \{1, 2\}$

* Random walk on random graphs: $t_N = \alpha \ln N$ (Lubetzky-Sly 09)

* Extension to finite quantum groups. (McCarthy 15-18)

* Some examples on infinite quantum groups (Freslon 18)



If there is $s_N = o(t_N)$ and f a continuous map non-increasing from 1 to 0 such that

$$d_N(t_N + c s_N) \xrightarrow[N\to\infty]{} f(c), \quad c \in \mathbb{R}$$

we say f is a <u>cutoff profile</u>

Rmk. Cutoff profile $\Rightarrow$ cutoff

Rmk. A cutoff profile is only unique up to affine transformation

$$f \sim f' \iff f = f'(a \cdot + b), \text{ for some } a>0,\ b \in \mathbb{R}$$

$$(t_N, s_N) \sim (t_N', s_N') \iff \begin{cases} t_N - t_N' = O(s_N) & \text{for some } a>0 \\ s_N = a s_N' \end{cases}$$

Some notable cutoff profiles

Rmk. $f(c) = d_{TV}(\eta_c, \mathrm{Haar}_\infty)$

* Random transpositions on S_N, $\begin{cases} t_N = \frac{1}{2} N \ln N \\ s_N = N \end{cases}$ $f(c) = d_{TV}(\mathrm{Poi}(1+e^{-c}), \mathrm{Poi}(1))$

(Teyssier 19)

* Random transpositions on S_N^+ (Freslon, Teyssier, Wang 22)
Brownian motion on O_N^+ / S_N^+

η_c has an atom when $c \leq 0$

* Brownian motion on U_N^+ (D-24)

η_c has a complex non-abs continuous part when $c \leq 0$

* Brownian motion on H_N^{s+} (D-24 or 25)? To come out soon.

$$d_{TV}(\eta_c, \nu_\infty)$$

has atom

II The unitary quantum group

We call unitary quantum group (of size N) the $*$-algebra $C(U_N^+)$ generated by N^2 elements $(u_{ij})_{1\leq i,j\leq N}$ such that

(i) $\sum_k u_{ik}u_{jk}^* = \delta_{ij} = \sum_k u_{ki}^* u_{kj}$ $\quad 1\leq i,j\leq N$ $\quad (u = (u_{ij})$ unitary$)$

(ii) $\sum_k u_{ik}^* u_{jk} = \delta_{ij} = \sum_k u_{ki} u_{kj}^*$ $\quad 1\leq i,j\leq N$ $\quad (u^t = (u_{ji})$ unitary$)$

It is equipped with a <u>coproduct</u> $\Delta : C(U_N^+) \to C(U_N^+)\otimes C(U_N^+)$ satisfying

$$\Delta(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}$$

It plays the role of the product.

Rmk. $S : C(U_N^+) \to C(U_N^+)^{op}$, $u_{ij}\mapsto u_{ji}^*$ antipode (inverse map)
$\varepsilon : C(U_N^+) \to \mathbb{C}$, $u_{ij}\mapsto \delta_{ij}$ counit (unit map).

Describing U_N^+'s representation theory using O_N^+'s $\left\{\begin{array}{l}\text{orthogonal} \\ \text{quantum group}\end{array}\right.$

$\to$ $C(O_N^+) \simeq C(U_N^+)/\langle u_{ij} - u_{ij}^* \rangle$ (denote by o_{ij} the image of u_{ij} through the quotient map $C(U_N^+) \to C(O_N^+)$).

Thm (Banica). The irreducible characters of O_N^+ may be labelled as $(\chi_n)_{n\in\mathbb{N}}$ with

$$\chi_n = P_n(\chi_1) \quad \text{where} \left\{\begin{array}{l} \chi_1 = \sum_i o_{ii} \\ \\ P_0 = 1,\ P_1 = X,\quad XP_n = P_{n+1} + P_{n-1},\ n \geq 1 \end{array}\right.$$

Thm (Banica). If z denotes the identity map on $\mathbb{T}$, then the map $u_{ij} \mapsto o_{ij} z$ extends to an isomorphism of quantum groups between $C(U_N^+)$ and its image in $C(O_N^+) * C(\mathbb{T})$.

Moreover, the irreducible characters of U_N^+ are the elements

$$\hat{\chi}_\emptyset = 1, \qquad \chi_{\underline{n}}^{\varepsilon} = z^{[\varepsilon_1]_-} \chi_{n_1} z^{\varepsilon_2} \dots z^{\varepsilon_p} \chi_{n_p} z^{[\varepsilon_{p+1}]_+}$$

$$\underline{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p, \quad \varepsilon = \pm 1,$$

$$\varepsilon_1 := \varepsilon, \quad \varepsilon_{i+1} = \varepsilon_i (-1)^{n_i+1}, \quad [\eta]_- = \min(\eta, 0), \quad [\eta]_+ = \max(\eta, 0)$$

Some examples : • $\hat{\chi}_1^+ = \chi_1 z = \sum_i u_{ii}$

• $\hat{\chi}_1^- = z^{-1}\chi_1 = \sum_i u_{ii}^*$

• $\hat{\chi}^+_{(2,1)} = \chi_2 z^{-1} \chi_1$

• $\chi_1 \chi_2 z^{-1}$ is not irredecible

Let $C(U_N^+)_0$ the <u>central algebra</u> that is the $*$-algebra generated by the characters.

$\longrightarrow$ There is a conditionnal expectation $\mathbb{E} : C(U_N^+) \to C(U_N^+)_0$.

$$\mathbb{E} : C(G) \to C(G),\ f \mapsto \{x \mapsto \int f(gxg^{-1})dg\}$$

Probability on U_N^+.

Def. A <u>state</u> on U_N^+ is a linear map $\varphi : \mathcal{O}(U_N^+) \to \mathbb{C}$ that is positive ($\varphi(aa^*) \geq 0 \ \forall a \in \mathcal{O}(U_N^+)$) and unital ($\varphi(1) = 1$)

Thm. There is a unique state h on U_N^+ that satisfies

$$h * \varphi = h = \varphi * h, \text{ for any state } \varphi \text{ on } U_N^+.$$

It is called the <u>Haar state</u> $\quad (\varphi * \psi := (\varphi \otimes \psi) \circ \Delta)$

On a classical compact group	Lévy process on U_N^+ $(\varphi_t)_{t \geq 0}$
(i) $\mu_0 = \delta_e$	(i) $\varphi_0 = \varepsilon : u_{ij} \mapsto \delta_{ij}$
(ii) $\mu_t \to \mu_0$ weakly as $t \to 0$	(ii) $\varphi_t \to \varphi_0$ as $t \to 0$ weakly
(iii) $\mu_{t+s} = \mu_t * \mu_s$	(iii) $\varphi_{t+s} = \varphi_t * \varphi_s$
(h is central)	$\varphi \circ \mathbb{E} = \varphi \iff \varphi$ is central

$\rightsquigarrow$ All the information of a Lévy process (φ) is contained within its generating functionnal

$$\mathcal{L} = \lim_{t \to 0} \frac{\varphi_t - \varepsilon}{t}.$$

III Brownian motion

We want to define the Brownian motion as an $\mathbb{E}$-invariant generating functional on $G(U_N^+)$.

Thm (Liao 04).

$$L = -b\Delta - \text{Lévy} \quad (b \geq 0) \text{ for any central generating functional}$$

Thm (Cipriano, Franz, Kula 13).

$$L : \begin{cases} G(O_N^+)_0 \longrightarrow \mathbb{C} \\ P(\chi_1) \longmapsto -bP'(N) - \int_{-N}^{N} \frac{P(N)-P(x)}{N-x} d\nu(x) \end{cases}$$

$O(O_N^+)_0 \simeq \mathbb{C}[X]$ | Rmk. Such a decomposition exists for S_N^+

Pbm. $G(U_N^+)_0 = \mathbb{C}\langle \chi_1, z, z^{-1}\chi_1 \rangle$

Solution → Centralized Gaussian processes are Brownian motion
→ Looking at the Brownian motion on U_N

$$O(S_N^+) = \mathbb{C}[\chi_0 + \chi_1]$$

$$L : P(\chi_0 + \chi_1) \mapsto -bP'(N)$$

IV Computing

Thm (D, 2024). We call Brownian motion of parameter (α, β) with $\alpha \geq \beta \geq 0$ on U_N^+, the central generating functionnal $\mathcal{L} : \mathcal{O}(U_N^+) \to \mathbb{C}$ defined by

$$\mathcal{L}(\chi_{\underline{n}}^{\varepsilon}) = -(\alpha - \beta) P_{\underline{n}}'(N) + \beta \frac{P_{\underline{n}} - 2E_{\underline{n}}}{N} P_{\underline{n}}(N)$$

where $P_{\underline{n}} = P_{n_1} \dots P_{n_p}$ $\quad (P_0 = 1,\ P_1 = X,\ XP_n = P_{n+1} + P_{n-1})$

$$\varphi_t(\chi_{\underline{n}}^{\varepsilon}) = P_{\underline{n}}(N) \exp\left(-t \, \mathcal{L}(\chi_{\underline{n}}^{\varepsilon}) / P_{\underline{n}}(N)\right)$$

Furthermore, the associated Lévy process has cutoff at time $t_N = \alpha N \ln N$. Moreover, we have partial cutoff profile, more precisely,

$$d_N\left(N \ln(\sqrt{2}N) + cN\right) \xrightarrow[N \to \infty]{} d_{TV}(\eta_c, \nu_{sc}), \qquad c > 0$$

$$\limsup_{N \to \infty} d_N\left(N \ln(\sqrt{2}N) + cN\right) \geq d_{TV}(\eta_c, \nu_{sc}), \quad c < 0$$

where ν_{sc} is the semi circular distribution

$$\nu_{sc}(P_m \cdot P_n) = \delta_{mn}$$

η_c the only distribution satisfying $\eta_c(P_n) = e^{-nc}$

Sketch of proof. Set $t_c := N\ln(\sqrt{2}N) + cN$

◇ Idea through moment convergence

$$\varphi_{t_c}(\chi_{\underline{n}}^{\varepsilon}) \xrightarrow[N\to+\infty]{} \exp(-\tilde{c}\,|\underline{n}|) \qquad \begin{cases} \tilde{c} = c + \ln\sqrt{2} \\ |\underline{n}| = n_1 + \dots + n_p \end{cases}$$

↪ what matters is the <u>composition type</u> $|\underline{n}|$ of $\underline{n}$

◇ Restricting to a smaller algebra.

Denote by x_m the sum of all characters whose tuple are of type m

$$x_0 = \hat{\chi}_{\emptyset} \;;\; x_1 = \hat{\chi}_1^+ + \hat{\chi}_1^- \;;\; x_2 = \hat{\chi}_{11}^+ + \hat{\chi}_2^+ + \hat{\chi}_{11}^- + \hat{\chi}_2^-$$

$$x_3 = \hat{\chi}_{111}^+ + \hat{\chi}_{12}^+ + \hat{\chi}_{21}^+ + \hat{\chi}_3^+ + \dots$$

The irreducible characters form an orthonormal family for $\langle \cdot, \cdot \rangle = h(\cdot^* \cdot)$

$$\tilde{x}_m := x_m / \sqrt{2}^m \;;\; \Longrightarrow \tilde{x}_m = P_m(\tilde{x}_1)$$

Construct an h-invariant conditional expectation

$$\mathbb{E} : \mathcal{O}(U_N^+)_\sigma \to \mathcal{O}(U_N^+)_{00} = \mathbb{C}[\hat{x}_1]$$

◇ Compute the limit profile on $O(U_N^+)_\infty$.

$$\hat{\varphi}_t = \varphi_t \circ \mathbb{F} \ (\neq \varphi_t)$$

Look at $\hat{\varphi}_t$ and h as probabilities through

$$\begin{cases} \mathbb{C}[\hat{x}_1] \longrightarrow \mathbb{C}[X] & \rightsquigarrow \quad \mu_t \\ \quad \hat{x}_1 \longmapsto X & \qquad \nu_{SC} \end{cases}$$

$\mu_{t_c} \xrightarrow[N \to \infty]{} \eta_c$ in moments $\left(\eta_c(P_m) = e^{-mc}\right)$

$$\hookrightarrow \ d_{TV}(\mu_{t_c}, \nu_{SC}) \longrightarrow d_{TV}(\eta_c, \nu_{SC})$$

◇ Finishing up on $O(U_N^+)_0$

$$d_{TV}(\varphi_{t_c}, \hat{\varphi}_{t_c}) \longrightarrow 0 \qquad c > 0$$

V / Some furthe questions

(1) Is the distance d_{TV} really interesting for quantum groups?
$\to$ Absolute continuity is easily lost in that case and this distance has no subtleties.

(2) What is interesting in a cutoff profile
$\to$ The profile f?
$\to$ The sequence (t_N, s_N)?

(3) To what extent does the profile f depends on the process? It seems the profile is strongly affected by the group's representation theory.

O_N^+ $\quad \varphi_{t_c}(P_m) \to e^{-m\alpha}$

Thank you!